CS 598:
AI Methods for Market Design

Lecture 4: Auction Design

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Announcements

- Two paper presentations today!
  - Get ready with the peer evaluation form
- HW1 will be out next week
  - You have 2 weeks to work on it and can work in pairs or individually

Outline

- Basic settings
- Characterizing sealed-bid auctions
- Second-price, sealed bid auction
- First-price, sealed bid auction
- Revenue equivalence
- Multi-round auctions

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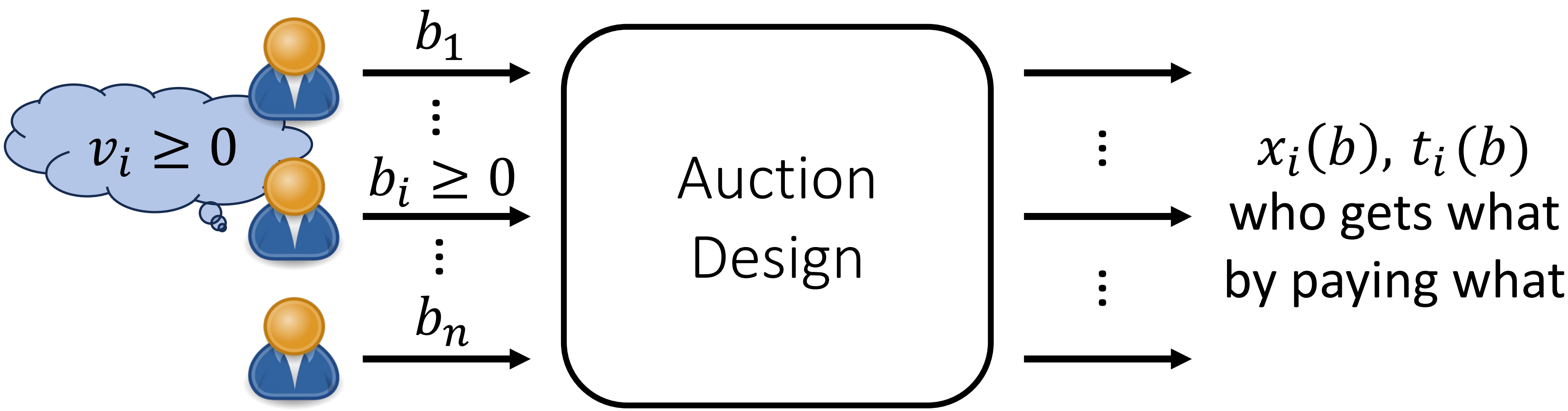
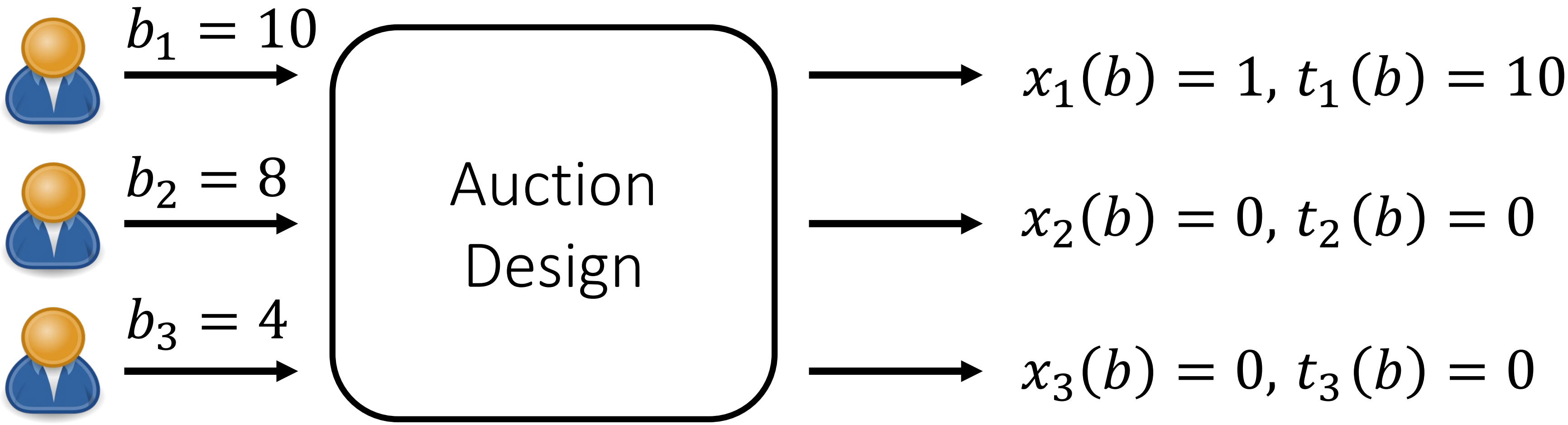
Basic Setting: Sealed-Bid Auctions



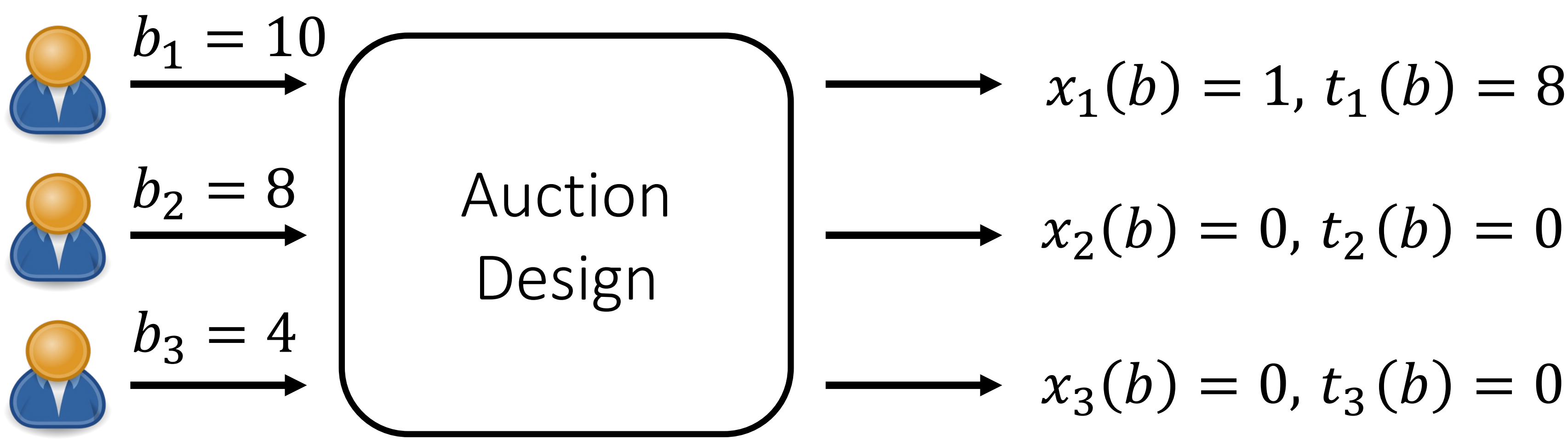
- A set of n bidders interested in buying a *single* item
- Each bidder submits a bid b_i without knowing other bids
- A bid profile: $b = (b_1, \dots, b_n)$
- An allocation rule: $x(b) \in \{0,1\}^n$
  - $x_{\mathrm{i}}(b) \in \{0,1\}$: whether bidder i is allocated the item
- A payment rule: $t(b) \in \mathbb{R}^n$
  - $t_i(b) \in \mathbb{R}$: payment made by bidder i

First-Price Sealed-Bid Auction



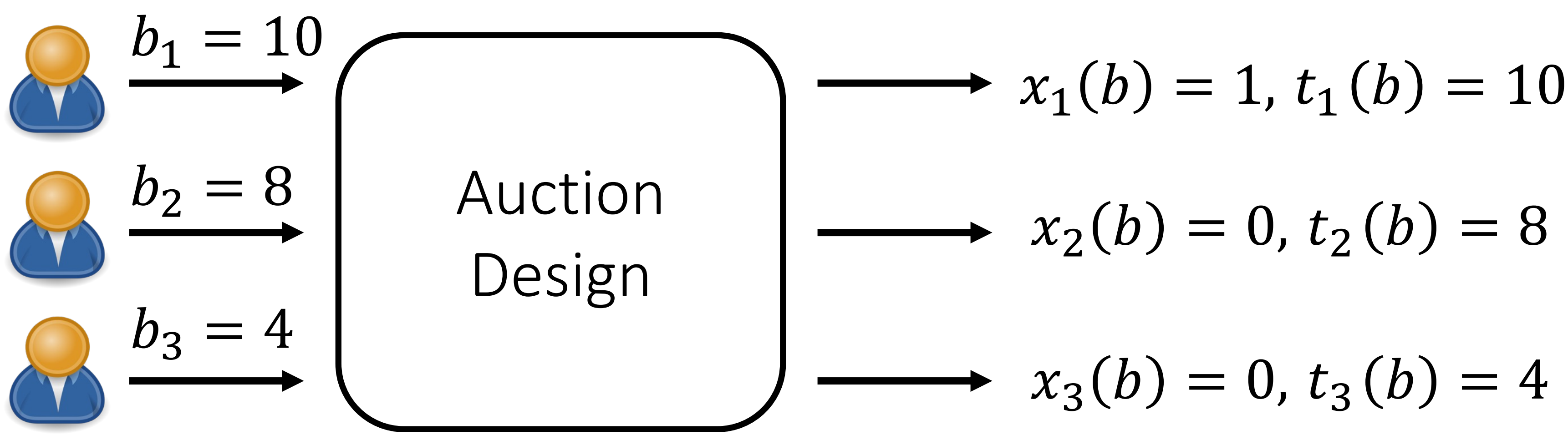
Allocate the item to bidder 1 and collect b_1 as the payment from bidder 1, with 0 payment from others.

Second-Price Sealed-Bid Auction



Allocate the item to bidder 1 and collect b_2 as the payment from bidder 1, with 0 payment from others.

All-Pay Auction



Allocate the item to bidder 1 and collect b_i as the payment from bidder i, regardless of winning or not.

Design Objectives

- Allocative efficiency (efficient auction)

Allocate the item to the bidder with the highest value

- Revenue maximization (optimal auction)

Maximize the expected revenue to the seller, across all possible auction rules

Models of Bidders' Values

- Private-value model
  - Value for upgrading to first class on a flight

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- Common-value model

Models of Bidders' Values

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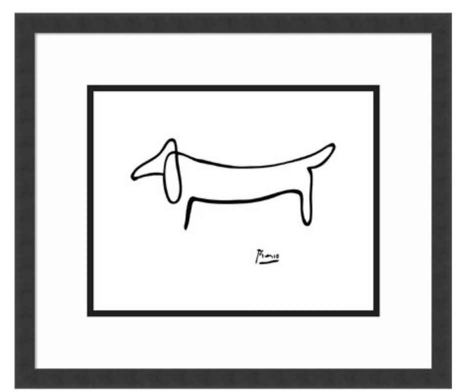
- Common-value model

- Interdependent value model



Models of Bidders' Values

Quasi-Linear Utility:

Given bid profile b, the utility of bidder i for allocation $x_i(b) \in \{0, 1\}$ and payment $t_i(b) \in \mathbb{R}$ is

$$u_i(b) = x_i(b) \cdot v_i - t_i(b)$$

- Utility depends linearly on the payment
- Value as willingness-to-pay, with no budget constraint

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Sealed-Bid Auction as a Game of Incomplete Information

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- Bidder strategy (sealed-bid auctions)

A strategy for bidder i, $s_i: R_{\geq 0} \rightarrow R_{\geq 0}$ defines a bid $s_i(v_i)$ for every possible value v_i of the bidder

- Strategy profile

$s(v) = (s_1(v_1), \ldots, s_n(v_n))$. Similarly, we have $s_{-i}(v_{-i})$

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- Dominant-strategy equilibrium (sealed-bid auctions)

Strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a DSE *if and only if*
$u_i\big(s_i^*(v_i), s_{-i}(v_{-i})\big) \geq u_i\big(b_i, s_{-i}(v_{-i})\big) \quad \forall i, v_i, b_i, v_{-i}, s_{-i}$

Sealed-Bid Auction as a Game of Incomplete Information

- Strategy-proof auction

A sealed-bid auction is strategy-proof if truthful bidding is a dominant-strategy equilibrium

Sealed-Bid Auction as a Game of Incomplete Information

- Efficient auction

A sealed-bid auction is efficient if, for every value profile v, the item is allocated in *an equilibrium* of the auction to the bidder with the highest value

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Second-Price Sealed-Bid Auction

Theorem. The second-price sealed bid auction is strategy-proof and efficient.

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Proof (strategy-proof / truthful):

Let $b' \coloneqq \max_{j \neq 1} s_j(v_j)$ be the highest bid from other bidders.

(1) If $v_1 > b'$, bidder 1's best response is $b_1 > b'$

(2) If $v_1 = b'$, bidder 1 is indifferent across all bids (0 utility)

(3) If $v_1 < b'$, bidder 1's best response is $b_1 < b'$

In every case, the truthful bid $v_1 = b_1$ is a best response.

Therefore, truthful bidding is a DSE.

Second-Price Sealed-Bid Auction

Theorem. The second-price sealed bid auction is strategy-proof and efficient.

Proof (efficient):

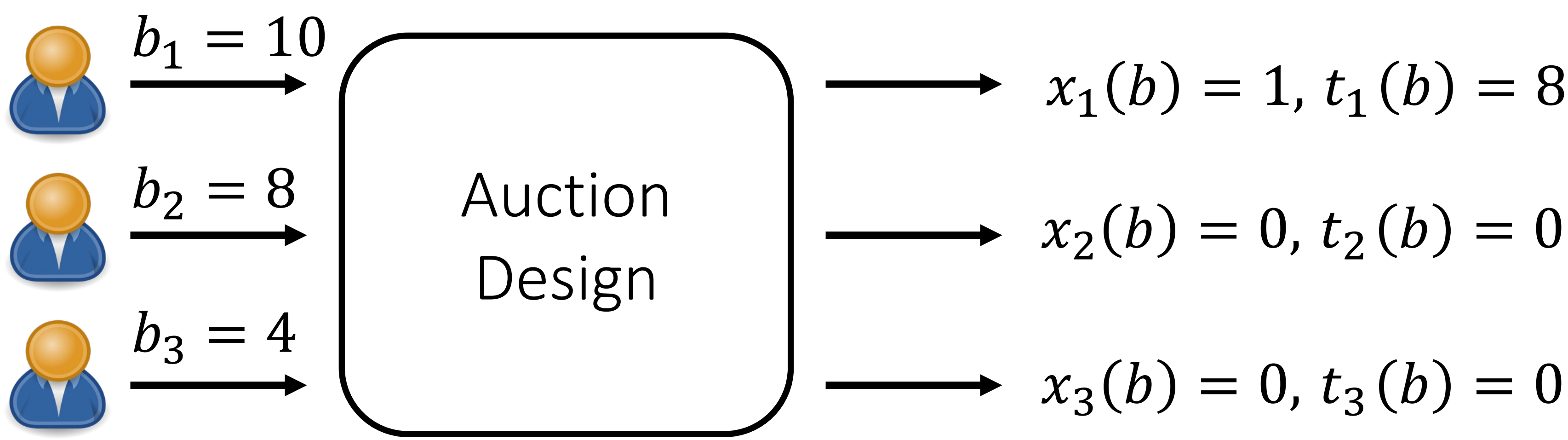
Since truthful bidding is a DSE, in this equilibrium, no bidder has a higher bid than the winner who has the highest value.

Second-Price Sealed-Bid Auction

Some more comments on SPSB auctions:

- Truthful bidding is preferred to sitting out
- Given a bid vector, computing the winner and the price can be very efficient
- Transparency and credibility: you trust the auctioneer on what the second highest bid is

Second-Price Sealed-Bid Auction



Allocate the item to bidder 1 and collect b_2 as the payment from bidder 1, with 0 payment from others.

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First-Price Sealed-Bid Auction

- Three bidders with i.i.d. private values uniform on (0, 12)
- How much will you bid? Will you stick to truthful bidding? Why?

First-Price Sealed-Bid Auction

- Model the beliefs where bidders have to reason about each other's values
- We have $v_i \sim G_i$, which is bidder i's cumulative distribution function
- We assume *common knowledge* on $G_1, \ldots, G_n$
- We assume bidders are risk neutral

First-Price Sealed-Bid Auction

- Bayesian Nash equilibrium (BNE):

A strategy profile that maximizes the expected payoff for each agent, given their beliefs and the strategies played by the other agents

- Strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a BNE in a sealed-bid auction if

$$\mathbf{E}_{v_{-i}}\left[u_i(s_i^*(v_i), s_{-i}^*(v_{-i}))\right] \geq \mathbf{E}_{v_{-i}}\left[u_i(b_i, s_{-i}^*(v_{-i}))\right]$$

$$\forall i, v_i, b_i$$

First-Price Sealed-Bid Auction

- Order-preserving strategy profile

A strategy profile in a sealed-bid auction is *order-preserving* if, for all bidders $i, j \in N$,

$$s_i(v_i) > s_j(v_j) \text{ if and only if } v_i > v_j$$

First-Price Sealed-Bid Auction

Theorem. For bidders with i.i.d. values uniform on [0, 1], the BNE in the FPSB auction is for each bidder to play the following strategy

$$s_i^*(v_i) = \left(\frac{n-1}{n}\right) v_i$$

Proof:

First-Price Sealed-Bid Auction

Theorem. The FPSB auction is efficient.

Proof:

The equilibrium strategy is order-preserving, so the bidder with the highest value has the highest bid.

First-Price Sealed-Bid Auction

Some more comments on FPSB auctions:

- BNE: Trade off between the risk of paying more than is necessary and losing by bidding too little
- Increasing the number of participants to increase revenue
- Rely on the rational assumption and common knowledge of the distribution on values
- The equilibrium behavior of an FPSB changes when the bids of others are known (Ad auction example)

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Which auction will have higher revenue?

Toy Example: Two Bidders

- The expected value of the k-th highest value on n independent samples drawn from $U(0,1)$:

$$\mathbf{E}\left[Z_{(k)} \mid n \text{ samples IID } \sim \ U(0,1)\right] = \frac{n-(k-1)}{n+1}$$

- FPSB: $E[v_{(1)}] = 2/3$, and thus $\text{Rev} = E[b_{(1)}] = 1/3$
- SPSB: $\text{Rev} = E[v_{(2)}] = 1/3$

Characterizing BNE in Auctions

Given an auction and an equilibrium strategy s^*, at an intermediate state where bidder i knows v_i

- Interim allocation for bidder i with v_i

$$x_i^*(v_i) = \mathbf{E}_{v_{-i}}\left[x_i(s_i^*(v_i), s_{-i}^*(v_{-i}))\right]$$

- Interim payment for bidder i with v_i

$$t_i^*(v_i) = \mathbf{E}_{v_{-i}}\left[t_i(s_i^*(v_i), s_{-i}^*(v_{-i}))\right]$$

- Interim utility then is $u_i^*(v_i) = v_i x_i^*(v_i) - t_i^*(v_i)$

Characterizing BNE in Auctions

Theorem. In any BNE of any sealed-bid auction, for bidder i with v_i, we have

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Proof: Given the definition of BNE, we have

$v_i x_i^*(v_i) - t_i^*(v_i) \geq v_i x_i^*(v_i') - t_i^*(v_i')$ for true value v_i

$v_i' x_i^*(v_i') - t_i^*(v_i') \geq v_i' x_i^*(v_i) - t_i^*(v_i)$ for true value v_i'

⬇

$$(v_i - v_i')(x_i^*(v_i) - x_i^*(v_i')) \geq 0$$

Characterizing BNE in Auctions

Theorem. In any BNE of any sealed-bid auction, for bidder i with v_i, we have

- Interim payment identity: For value v_i and interim allocation $x_i^*(v_i)$, the interim payment is

$$t_i^*(v_i) = v_i \times x_i^*(v_i) - \int_{z=v_{\min}}^{v_i} x_i^*(z)\mathrm{d}z - C_i$$

where C_i is a constant.

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Proof:

Rate of increase in interim payment = Rate of increase in interim value

Characterizing BNE in Auctions

Example: Three bidders with i.i.d. values from $U(0,1)$

The interim allocation:

$$x_1^*(v_1) = P(\max(v_2, v_3) \leq v_1) = v_1^2$$

The interim payment for FPSB, SPSB, all-pay auction is:

$$t_1^*(v_1) = v_1 x_1^*(v_1) - \int_{z=0}^{v_1} x_1^*(z)\mathrm{d}z = v_1 \cdot v_1^2 - \int_{z=0}^{v_1} z^2 \mathrm{d}z = v_1^3 - \frac{1}{3}v_1^3 = \left(\frac{3-1}{3}\right) v_1^3$$

N-bidder case:

$$s_i^*(v_i) = \left(\frac{n-1}{n}\right) v_i^n$$

Which auction will have higher revenue?

Revenue Equivalence

A *normalized auction* is one where a bidder with value 0 (or v_{min}) has zero interim utility

Theorem. Any two normalized, sealed-bid auctions that each have a BNE with an identical interim allocation have the same expected revenue in these two BNE.

Proof:

same interim allocation → same interim payment

→ same revenue

$$\mathrm{Rev} = \sum_{i=1}^{n} \mathbf{E}_{v_i}[t_i^*(v_i)]$$

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Some more notes on revenue equivalence:

- From the theorem, we have that all efficient auctions have the same expected revenue

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Revenue Equivalence

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- For i.i.d. private value environments with bounded support, the FPSB auction has a symmetric, increasing, and unique BNE

- For IID private value environments with bounded support, the expected revenue of the FPSB auction is the same as the expected revenue in the truthful DSE of the SPSB auction
  - The interim allocation is the same for the BNE of FPSB and the truthful DSE of the SPSB auction

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- Is there a BNE in a SPSB auction?

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- Is there a BNE in a SPSB auction?

$$s_1^*(v_1) = 1, \quad s_2^*(v_2) = 0$$

Revenue Equivalence

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- Is there a BNE in a SPSB auction?

$$s_1^*(v_1) = 1, \quad s_2^*(v_2) = 0$$

- Is FPSB or SPSB auction optimal?

Finding a BNE in an Auction: Guess & Verify

- Guess that some auction design, A, with values i.i.d. sampled from distribution G has an efficient BNE and is normalized (i.e., $s_i(v_i) = 0$ for $v_i = 0$)
- Construct a strategy profile s, such that the interim payment in auction A at s is equal to the truthful DSE of a SPSB auction
- Verify that s is a BNE in auction A. Confirm that auction A is efficient with s and normalized

Finding a BNE in an Auction: Guess & Verify

Question:

Follow guess & verify to derive the BNE for (1) the all-pay auction and (2) the FPSB auction

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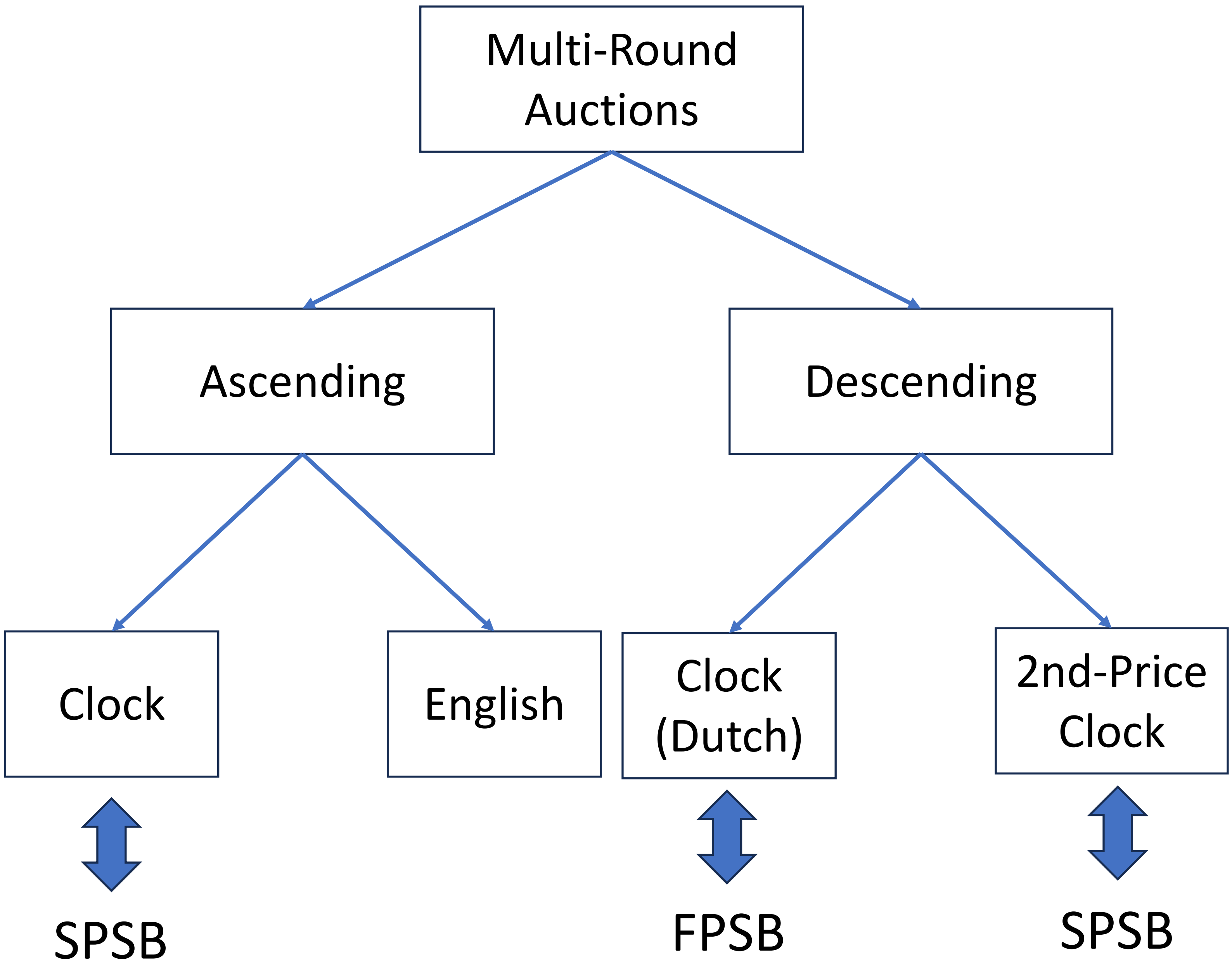
Multi-Round Auctions

- May allow bidders to respond to others' bids, esp. in interdependent or common value scenarios
- May have more flexible strategies
- Can be helpful in transparency and credibility


Multi-Round Auctions
Ascending
Descending
Clock
English
Clock (Dutch)
2nd-Price Clock
SPSB
FPSB
SPSB